OPTIMAL FIXED-POINT CONTINUOUS LINEAR SMOOTHING

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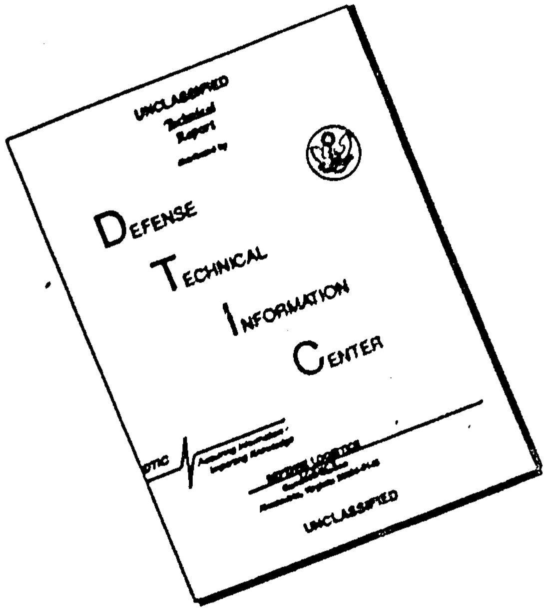

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ABSTRACT

The filter and error covariance equations for optimal fixed-point smoothing for continuous linear systems are developed. The development is carried out by considering the limiting case of the results for the same problem for discrete linear systems. The procedure is of use in estimation problems where a smoothed estimate of a continuous linear system's state is desired at some specified critical time during the system's operation. Four examples are presented to illustrate the results.

1.0 Introduction

The problem of prediction and filtering for continous linear systems has been treated in considerable detail by Kalman and Bucy[1]. The problem of smoothing for continous linear systems for the case where the measurement interval $t_o \leq t \leq \tau$ (t = variable time) is fixed and a smoothed estimate of the system's state x is obtained for all $t \epsilon [t_o, \tau]$ has been solved by Bryson and Frazier[2], and Rauch, Tung, and Striebel[3]. The smoothed estimate is the solution of the system of equations

$$\dot{\hat{x}}(t/\tau) = F(t)\,\hat{x}(t/\tau) + Q(t)P^{-1}(t)\,[\hat{x}(t/\tau) - \hat{x}(t)] \qquad t_o \leq t \leq \tau \tag{1}$$

where $\hat{x}(t/\tau)$ is the smoothed estimate, an n-vector; $F(t)$ is a continous $n \times n$ matrix which is defined by the plant dynamics; $Q(t)$ is the $n \times n$ positive semidefinite covariance matrix of the zero mean, Gaussian white noise plant disturbance; $P^{-1}(t)$ is the inverse of the $n \times n$ filtering error covariance matrix; $\hat{x}(t)$ is the filtered estimate of x at time t; and the dot denotes the derivative with respect to t. Equation (1) is subject to the boundary condition $\hat{x}(\tau/\tau) = \hat{x}(\tau)$ which is obtained from the filtering solution. Hence, $\hat{x}(t/\tau)$ is obtained by integrating Eq. (1) backward in time from τ.

If a smoothed estimate of x is desired at only one value of $t \epsilon [t_o, \tau]$, say at some critical time during the plant's operation, the above smoothing procedure is inefficient from a computational point of view. It would be more desirable to have a smoothing algorithm that begins with an initial estimate of $x(t)$ and updates this estimate by recursively processing the measurements in the interval $[t, \tau]$. Thus the smoothed estimate could be "built-up" as more measurement data become available, rather than by "back-tracking" from $\hat{x}(\tau)$ as required by Eq. (1). Additionally, if the terminal time τ is not fixed or specified a priori, as might be the case in an "on-line" smoothing problem, the smoothing formulation of Eq. (1) is not applicable. The case where t is fixed, and τ is either fixed or free, $t \leq \tau$, is termed the fixed-point smoothing problem, and is the subject of this paper.

This problem arises in such physical situations as: (1) assessing the performance of a midcourse space guidance system from telemetry and tracking data taken after the midcourse maneuvar is completed, and (2) determining the errors in a telemetry or communication channel at some known critical time from measurements and recordings taken after that time. In both of these cases, the terminal measurement time may not be known a priori, but might be governed by such factors as fading and interference during the experiment.

In this paper, the fixed-point smoothing problem for continous linear systems will be treated by considering the limiting case of the fixed-point smoothing solution for discrete linear systems for which the solution is known[4, 5]. The limiting procedure to be used is due to Kalman[6].

Four examples are included to illustrate the results.

2.0 Fixed-Point Discrete Linear Smoothing[4, 5]

Since the relations for fixed-point continous linear smoothing will be developed by considering the limiting case of the fixed-point discrete linear smoothing solution, the results for the latter are summarized below.

Consider the discrete linear system

$$x(k+1) = \Phi(k+1, k)\, x(k) + w(k) \tag{2}$$

$$z(k+1) = H(k+1)\, x(k+1) + v(k+1) \tag{3}$$

where x is an n-vector, the state; z is an m-vector, the measurement; Φ is a real nxn matrix, the state transition matrix; H is a real mxn matrix; and $k = 0, 1, \ldots,$ is the discrete time index. In addition, w and v are independent, zero mean, Gaussian white sequences for which

$$E[w(j)\, w'(k)] = Q(k)\, \delta_{jk}$$

$$E[v(j)\, v'(k)] = R(k)\, \delta_{jk}$$

where E denotes the expected value, the prime denotes the matrix transpose, and δ_{jk} is the Kronecker delta. Here, Q(k) is a real nxn positive semidefinite matrix and R(k) is a real mxm positive definite matrix. The initial state x(0) is assumed to be a zero mean, Gaussian random n-vector which is independent of w and v for all k, and for which

$$E[x(0)\, x'(0)] = P(0)$$

where P(0) is a real nxn positive semidefinite matrix.

Let $\hat{x}(k/n)$, $k < N$, denote a smoothed estimate of x at time k based on measurements up to and including the one at time N. Also, let the smoothing error be defined by the relation

$$\tilde{x}(k/N) = x(k) - \hat{x}(k/N)$$

and the mean square smoothing error by the expression

$$\mathcal{E} = E[\tilde{x}'(k/N)\,\tilde{x}(k/N)]$$

Then, it can be shown[4, 5] that the smoothed estimate that minimizes $\mathcal{E}$ (termed the optimal smoothed estimate in the sequel) is given by the recursive relation

$$\hat{x}(k/N) = \hat{x}(k/N-1) + K(N-1, k)[\hat{x}(N) - \hat{x}(N/N-1)] \tag{4}$$

for N = k+1, k+2, . . . where

$$K(N-1, k) = \prod_{i=k}^{N-1} J(i) \tag{5}$$

$$J(i) = P(i)\,\Phi'(i+1, i)\,M^{-1}(i+1) \tag{6}$$

k = integer = constant, $[\ ]^{-1}$ denotes the matrix inverse and $\hat{x}(N)$ and $\hat{x}(N/N-1)$ are, respectively, the optimal filtered and predicted estimates of x at time N. The latter two estimates are governed by the set of relations[7]

$$\hat{x}(N/N-1) = \Phi(N, N-1)\,\hat{x}(N-1) \tag{7}$$

$$\hat{x}(N) = \hat{x}(N/N-1) + K^{o}(N)[z(N) - H(N)\,\hat{x}(N/N-1)] \tag{8}$$

$$K^{o}(N) = M(N)\,H'(N)\,[H(N)\,M(N)\,H'(N) + R(N)]^{-1} \tag{9}$$

$$M(N) = \Phi(N, N-1)\,P(N-1)\,\Phi'(N, N-1) + Q(N-1) \tag{10}$$

$$P(N) = [I - K^{o}(N)\,H(N)]\,M(N) \tag{11}$$

for N = 1, 2, . . . , where $\hat{x}(0) = 0$, P(0) is assumed given, and I is the n x n identity

matrix.

The initial condition for Eq. (4) is

$$\hat{x}(k/N-1)\Big|_{N = k+1} = \hat{x}(k/k) = \hat{x}(k) \tag{12}$$

which is obtained from Eq. (8) when N = k.

The nxn matrices M(N) and P(N-1), which also occur in Eq. (6), are the covariance matrices of the prediction error

$$\tilde{x}(N/N-1) = x(N) - \hat{x}(N/N-1)$$

and the filtering error

$$\tilde{x}(N-1) = x(N-1) - \hat{x}(N-1)$$

respectively.

The nxm matrix $K^o(N)$ is called the optimal filter gain and is also given by the relation

$$K^o(N) = P(N)\, H'(N)\, R^{-1}(N) \tag{13}$$

Similarly, the nxn matrix K(N-1, k) in Eq. (5) is termed the optimal smoothing filter gain.

The covariance matrix of the smoothing error for fixed-point discrete optimal linear smoothing is given by the first-order matrix difference equation[4, 5]

$$P(k/N) = P(k/N-1) + K(N-1, k)\, [P(N) - M(N)]\ K'(N-1, k) \tag{14}$$

for N = k+1, k+2, . . . , where the initial condition is

$$P(k/N-1)\Big|_{N = k+1} = P(k/k) = P(k) \tag{15}$$

which is obtained from Eq. (11) when N = k.

Finally, it should be noted from Eq. (5) that the smoothing filter gain matrix can be expressed by the recursive relation

$$K(N-1, k) = K(N-2, k)\ J(N-1) \tag{16}$$

$N = k+1, k+2, \ldots$.

3.0 Fixed-Point Continous Linear Smoothing

Consider the system of Eqs. (2) and (3) when the time between measurements is made arbitrarily small. Let the two time instants k and k+1 be replaced by t and $t + \Delta t$, respectively, where $\Delta t > 0$. Also, let the plant disturbance w(k) be replaced by $u(t)\ \Delta t$. Then, Eqs. (2) and (3) become

$$x(t + \Delta t) = \Phi(t + \Delta t, t)\ x(t) + u(t)\ \Delta t \tag{17}$$

and

$$z(t + \Delta t) = H(t + \Delta t)\ x(t + \Delta t) + v(t + \Delta t) \tag{18}$$

Assume that $\Phi(t + \Delta t, t)$ is the state transition matrix of the homogenous linear system

$$\dot{x} = F(t)\ x$$

where F(t) is a real continous $n \times n$ matrix. Then, $\Phi(t + \Delta t, t)$ can be expanded in a Taylor series to obtain

$$\Phi(t + \Delta t, t) = I + F(t)\ \Delta t + 0(\Delta t^2) \tag{19}$$

where $0(\Delta t^2)$ denotes terms of order $(\Delta t)^2$.

Substituting Eq. (19) into Eq. (17) and rearranging terms, there results

$$x(t + \Delta t) - x(t) = [F(t)\ x(t) + u(t)]\ \ \Delta t + 0(\Delta t^2)$$

Dividing through by Δt and taking lim $\Delta t \to 0$ gives

$$\dot{x} = F(t)\, x + u(t)$$

and taking lim $\Delta t \to 0$ in Eq. (18) leads to the relation

$$z(t) = H(t)\ x(t) + v(t)$$

In taking this limit, care must be exercised in defining the Gaussian white noise processes u(t) and v(t) as limits of the Gaussian white sequences w(k) and v(k), respectively. In particular, it has been shown[3, 6] that the covariance matrices Q(k) and R(k) must be replaced by $Q(t)\,\Delta t$ and $R(t)/\Delta t$, respectively, in all relations involving these covariance matrices in order that the description of the disturbances be physically meaningful in the limit. The details are given elsewhere[3, 6] and will not be repeated here.

The above limiting procedure will now be applied to develop the equations for fixed-point continous linear smoothing. Consider the smoothing interval [k, N] , and let the discrete time instants k, N-2, N-1, and N be denoted by T, $t - \Delta t$, t, and $t + \Delta t$, respectively, where $\Delta t > 0$. Then, Eq. (4) becomes

$$\hat{x}(T/t + \Delta t) - \hat{x}(T/t) = K(t, T)\,[\hat{x}(t + \Delta t) - \hat{x}(t + \Delta t/t)]$$

Dividing through by Δt and taking $\lim \Delta t \to 0$, it is seen that the fixed-point smoothed estimate must satisfy the system of ordinary linear differential equations

$$\dot{\hat{x}}(T/t) = \lim_{\Delta t \to 0} \frac{K(t, T)\,[\hat{x}(t + \Delta t) - \hat{x}(t + \Delta t/t)]}{\Delta t} \tag{20}$$

The limit on the right-hand side of Eq. (20) will now be evaluated. Consider first the expression for K(t, T) which, from Eqs. (16) and (6), is

$$\begin{aligned} K(t, T) &= K(t - \Delta t, T)\, J(t) \\ &= K(t - \Delta t, T)\, P(t)\, \Phi'(t + \Delta t, t)\, M^{-1}(t + \Delta t) \end{aligned} \tag{21}$$

It then follows that

$$\begin{aligned} \lim_{\Delta t \to 0} K(t, T) &= \lim_{\Delta t \to 0} K(t - \Delta t, T)\, P(t)\, \Phi'(t + \Delta t, t)\, M^{-1}(t + \Delta t) \\ &= \lim_{\Delta t \to 0} K(t - \Delta t, T)\, P(t)\, M^{-1}(t + \Delta t) \end{aligned} \tag{22}$$

Making the appropriate substitutions into Eq. (10), it is seen that

$$M(t + \Delta t) = \Phi(t + \Delta t, t)\, P(t)\, \Phi'(t + \Delta t, t) + Q(t)\, \Delta t \tag{23}$$

$$= [I + F(t)\,\Delta t + 0(\Delta t^2)]\, P(t)\, [I + F(t)\,\Delta t + 0(\Delta t^2)]' + Q(t)\,\Delta t$$

$$= P(t) + [F(t)\,P(t) + P(t)\,F'(t) + Q(t)]\,\Delta t + 0(\Delta t^2)$$

$$= P(t) + 0(\Delta t)$$

where $0(\Delta t)$ denotes terms of order Δt. It then follows that

$$P(t)\, M^{-1}(t + \Delta t) = P(t)\, [P(t) + 0(\Delta t)]^{-1}$$

$$= [I + 0(\Delta t)]^{-1}$$

Substituting this result into Eq. (21) yields

$$\lim_{\Delta t \to 0} K(t, T) = \lim_{\Delta t \to 0} K(t - \Delta t, T)\, [I + 0(\Delta t)]^{-1}$$

$$= K(t, T) \tag{24}$$

From Eqs. (7) and (8), the second factor in the limit on the right-hand side of Eq. (20) is given by the relation

$$\hat{x}(t + \Delta t) - \hat{x}(t + \Delta t/t) = K^{o}(t + \Delta t)\, [z(t + \Delta t) - H(t + \Delta t)\, \Phi(t + \Delta t, t)\, \hat{x}(t)] \tag{25}$$

From Eq. (13) and the fact that $R(N)$ must be replaced by $R(t + \Delta t)/\Delta t$,

$$K^{o}(t + \Delta t) = P(t + \Delta t)\, H'(t + \Delta t)\, R^{-1}(t + \Delta t)\, \Delta t \tag{26}$$

Substituting Eq. (26) into Eq. (25) and the result into Eq. (2) gives

$$\dot{\hat{x}}(T/t) = \lim_{\Delta t \to 0} \frac{K(t, T) P(t+\Delta t) H'(t+\Delta t) R^{-1}(t+\Delta t) [z(t+\Delta t) - H(t+\Delta t) \Phi(t+\Delta t, t) \hat{x}(t)]\, \Delta t}{\Delta t}$$

from which it follows immediately that

$$\dot{\hat{x}}(T/t) = K(t, T)\, P(t)\, H'(t)\, R^{-1}(t)[z(t) - H(t)\, \hat{x}(t)] \tag{27}$$

for $t \geqslant T$ where use has been made of the result in Eq. (24) in taking the limit. From Eq. (12), the initial condition on Eq. (27) is

$$\hat{x}(T/t)\Big|_{t = T} = \hat{x}(T/T) = \hat{x}(T)$$

There remains now the task of obtaining an algorithm for determining $K(t, T)$. First, it is seen that Eq. (21) can be written

$$K(t, T)\, J^{-1}(t) = K(t - \Delta t, T) \tag{28}$$

where

$$J^{-1}(t) = M(t + \Delta t)[\Phi'(t + \Delta t, t)]^{-1} P^{-1}(t)$$

which follows from Eq. (6). It is noted that $J^{-1}(t)$ exists if and only if $P(t)$ is nonsingular. If $P(t)$ is singular, it follows that $\hat{x}(t) = x(t)$ and there is no need for smoothing. Hence, it is assumed in the sequel that $P(t)$ is nonsingular.

Noting that $[\Phi'(t + \Delta t, t)]^{-1} = \Phi'(t, t + \Delta t)$, it follows from Eq. (23) that

$$M(t + \Delta t)\, \Phi'(t, t + \Delta t) = \Phi(t + \Delta t, t)\, P(t) + Q(t)\, \Phi'(t, t + \Delta t)\, \Delta t \tag{29}$$

The Taylor series expansion for $\Phi'(t, t + \Delta t)$ is

$$\Phi'(t, t + \Delta t) = I - F'(t)\, \Delta t + 0(\Delta t^2)$$

Substituting this result and Eq. (19) into Eq. (29) and grouping terms gives

$$\begin{aligned} M(t + \Delta t)\, \Phi'(t, t + \Delta t) &= [I + F(t)\, \Delta t + 0(\Delta t^2)]\; P(t) \\ &\quad + Q(t)[I - F'(t)\, \Delta t + 0(\Delta t^2)]\; \Delta t \\ &= P(t) + [F(t)\, P(t) + Q(t)]\, \Delta t + 0(\Delta t^2) \end{aligned}$$

Postmultiplying in this equation by $P^{-1}(t)$ then gives the result

$$J^{-1}(t) = I + [F(t) + Q(t)\, P^{-1}(t)]\, \Delta t + 0(\Delta t^2) \tag{30}$$

Substituting Eq. (30) into Eq. (28) yields

$$K(t, T) \left\{ I + [F(t) + Q(t)\, P^{-1}(t)]\, \Delta t + 0(\Delta t^2) \right\} = K(t - \Delta t, T)$$

which can also be written as

$$K(t, T) - K(t - \Delta t, T) = - K(t, T) \left\{ [F(t) + Q(t)\, P^{-1}(t)]\, \Delta t + 0(\Delta t^2) \right\}$$

Dividing through by Δt and taking $\lim \Delta t \to 0$ gives the $n \times n$ matrix linear ordinary differential equation

$$\dot{K}(t, T) = - K(t, T)[F(t) + Q(t)\, P^{-1}(t)] \tag{31}$$

where it is emphasized that the dot denotes the derivative with respect to t.

The initial condition for Eq. (31) is obtained by considering the time instants T and $T + \Delta T, \Delta T > 0$, and noting from Eqs. (5), (6), and (24) that

$$\begin{aligned}
K(T, T) &= \lim_{\Delta T \to 0} K(T, T) \\
&= \lim_{\Delta T \to 0} J(T) \\
&= \lim_{\Delta T \to 0} P(T)\, \Phi'(T + \Delta T, T)\, M^{-1}(T + \Delta T) \\
&= \lim_{\Delta T \to 0} P(T)\, M^{-1}(T + \Delta T) \\
&= \lim_{\Delta T \to 0} [I + 0(\Delta T)]^{-1} = I
\end{aligned}$$

Hence, the required initial condition for Eq. (31) is $K(T, T) = I$.

At this point, it is seen that Eqs. (27) and (31) along with their corresponding initial conditions specify the fixed-point smoothing filter for continous linear systems. However, in order to use this filter, it is necessary to have the filtered estimate $\hat{x}(t)$ for all $t \geqslant T$, and the filtering error covariance matrix $P(t)$ also for all $t \geqslant T$. These two quantities are obtained directly from the optimal filter solution for continous linear systems[1]:

$$\dot{\hat{x}} = F(t)\,\hat{x} + K^{o}(t)\,[z(t) - H(t)\,\hat{x}] \qquad t \geq t_o \tag{32}$$

$$\hat{x}(t_o) = 0$$

$$K^{o}(t) = P(t)\,H'(t)\,R^{-1}(t) \tag{33}$$

$$\dot{P} = F(t)\,P + P\,F'(t) - P\,H'(t)\,R^{-1}(t)\,H(t)\,P + Q(t) \tag{34}$$

$$P(t_o) \text{ given}$$

From Eq. (33), it is seen that Eq. (27) can also be written

$$\dot{\hat{x}}(T/t) = K(t, T)\,K^{o}(t)\,[z(t) - H(t)\,\hat{x}(t)] \tag{35}$$

for $t \geq T$.

The block diagram for the smoothing filter is shown in Fig. 1.

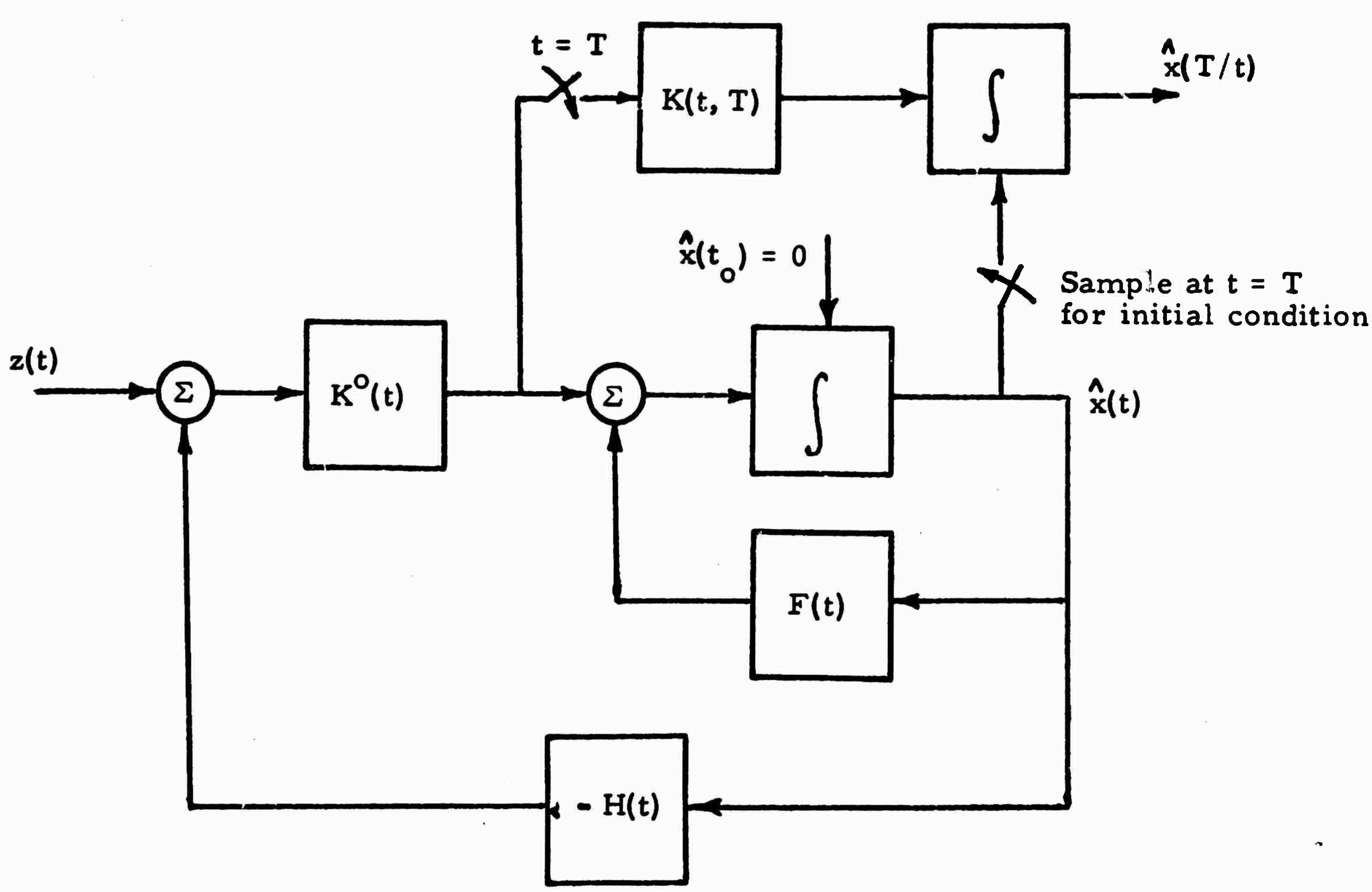


Fig. 1

Block diagram of fixed-point continous linear smoothing filter.

4.0 Fixed-Point Smoothing Error Covariance Matrix

A linear ordinary n x n matrix differential equation whose solution is the covariance matrix of fixed-point smoothing error will now be developed by applying the limiting processes of Section 3.0 to Eq. (14).

Replacing k by T, N-1 by t, and N by t + Δt in Eq. (14) yields

$$P(T/t + \Delta t) - P(T/t) = K(t, T)[P(t + \Delta t) - M(t + \Delta t)] K'(t, T)$$

Dividing through by Δt and taking lim Δt → 0 gives

$$\dot{P}(T/t) = \lim_{\Delta t \to 0} \frac{K(t, T)[P(t + \Delta t) - M(t + \Delta t)] K'(t, T)}{\Delta t} \tag{36}$$

From Eqs. (11) and (26),

$$\begin{aligned} P(t + \Delta t) - M(t + \Delta t) &= - K^{o}(t + \Delta t) H(t + \Delta t) M(t + \Delta t) \\ &= - P(t+\Delta t) H'(t+\Delta t) R^{-1}(t+\Delta t) M(t + \Delta t) \Delta t \end{aligned}$$

Substituting this result into Eq. (36), noting from Eq. (23) that

$$\lim_{\Delta t \to 0} M(t + \Delta t) = P(t)$$

and utilizing the result in Eq. (24), Eq. (36) simplifies to

$$\dot{P}(T/t) = - K(t, T) P(t) H'(t) R^{-1}(t) H(t) P(t) K'(t, T) \tag{37}$$

for $t \geqslant T$. Equation (36) can also be written

$$\dot{P}(T/t) = - K(t, T) K^{o}(t) H(t) P(t) K(t, T) \tag{38}$$

for $t \geqslant T$ where $K^{o}(t)$ is given by Eq. (33).

The initial condition for Eqs. (37) or (38) is

$$P(T/t)\Big|_{t = T} = P(T/T) = P(T)$$

which follows directly from Eq. (15).

The covariance matrix of the smoothing error is then the solution of either Eq. (37) or (38) with P(T) as the initial condition.

5.0 Examples

Example 1. Consider the scalar system

$$\dot{x} = u(t)$$

$$z(t) = x(t) + v(t)$$

for $t \geq 0$ where u(t) and v(t) are independent, zero mean, Gaussian white noise processes having constant variances σ_u^2 and σ_v^2, respectively, and x(0) is a zero mean, Gaussian random variable which is independent of u(t) and v(t) for all $t \geq 0$ and $E[x^2(0)] = \sigma_o^2$.

From Eq. (34), the variance of the filtered estimate of x is governed by the ordinary, first-order, scalar differential equation

$$\dot{P} = -\frac{1}{\sigma_v^2} P^2 + \sigma_u^2$$

for $t \geq 0$ where $P(0) = \sigma_o^2$. The solution to this differential equation is

$$P(t) = a \frac{1 + \mu e^{-\beta t}}{1 - \mu e^{-\beta t}} \qquad t \geq 0 \tag{39}$$

where

$$a = \sigma_v \sigma_u$$

$$\mu = \frac{\sigma_o^2 - a}{\sigma_o^2 + a}$$

and

$$\beta = \frac{2\sigma_u}{\sigma_v}$$

The optimal filter gain is seen to be

$$K^o(t) = P(t)\, H'(t)\, R^{-1}(t)$$

$$= \frac{a}{\sigma_v^2} \quad \frac{1 + \mu e^{-\beta t}}{1 - \mu e^{-\beta t}}$$

$$= \frac{\beta}{2} \quad \frac{1 + \mu e^{-\beta t}}{1 - \mu e^{-\beta t}}$$

Now consider the problem of smoothing in order to determine an estimate of the system's initial condition. It is noted that the initial estimate without smoothing has a variance of σ_o^2.

Substituting the appropriate quantities into Eq. (31),

$$\dot{K}(t, 0) = -\frac{\beta}{2} \quad \frac{1 - \mu e^{-\beta t}}{1 + \mu e^{-\beta t}} \quad K(t, 0)$$

where $K(0, 0) = 1$. The solution is obtained by direct integration and found to be

$$K(t, 0) = \frac{(1 + \mu)\, e^{-\frac{\beta t}{2}}}{1 + \mu e^{-\beta t}} \tag{40}$$

Substituting Eqs. (39) and (40) along with $H(t) = 1$ and $R^{-1}(t) = 1/\sigma_v^2$ into Eq. (37), it is seen that

$$\dot{P}(0/t) = -\sigma_u^2 \quad \frac{(1 + \mu)^2\, e^{-\beta t}}{(1 - \mu e^{-\beta t})^2} \tag{41}$$

where $P(0/0) = \sigma_o^2$.

After some manipulation, the solution of Eq. (41) is found to be

$$P(0/t) = \frac{a(1 + \mu)}{2} \qquad \frac{1 + e^{-\beta t}}{1 - \mu e^{-\beta t}}$$

for $t \geqslant 0$ and is, of course, the variance of the smoothed estimate of $x(0)$.

It is interesting to note the limiting behavior of the smoothing error variance. First, it is seen from its definition that $-1 \leqslant \mu \leqslant 1$. Hence, for $t \approx \frac{4}{\beta}$, i.e., for

a smoothing time of approximately four "time constants", $e^{-\beta t} = e^{-4} \approx 0.02$, and

$$P(0/t) \approx \frac{\alpha(1+\mu)}{2} = \frac{\sigma_o^2 \sigma_v \sigma_u}{\sigma_o^2 + \sigma_v \sigma_u}$$

Now suppose $\sigma_o^2 >> \sigma_v \sigma_u$. Then,

$$P(0/t) \approx \sigma_v \sigma_u << \sigma_o^2$$

and it is clear that a large reduction of the uncertainty associated with the estimate of x(0) has been effected by smoothing.

Next, consider the case when $\sigma_o^2 \approx \sigma_v \sigma_u$ where it is seen that

$$P(0/t) \approx \frac{\sigma_o^2}{2}$$

thereby giving a 50% reduction.

Finally, if $\sigma_o^2 << \sigma_v \sigma_u$,

$$P(0/t) \approx \sigma_o^2$$

and there is no reduction. In this case, smoothing is of no value, and the initial estimate $\hat{x}(0) = 0$ must be accepted as the optimal one.

For this example,

$$K(t, 0)\, K^o(t) = \frac{\beta}{2} \frac{(1+\mu)\, e^{-\beta t}}{1 - \mu e^{-\beta t}}$$

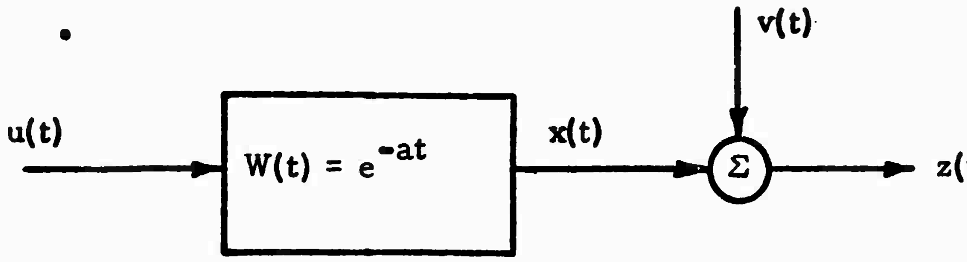
and Eq. (27) becomes

$$\hat{x}(0/t) = \frac{\beta}{2} \frac{(1+\mu)\, e^{-\beta t}}{1 - \mu e^{-\beta t}} \left[z(t) - \hat{x}(t) \right]$$

for $t \geq 0$ where the initial condition is $\hat{x}(0/0) = \hat{x}(0) = 0$.

Example 2. A problem commonly treated using Wiener filter theory(8, 9) is the

one for which the block diagram is shown below. Here, u(t) and v(t) are indepen-



$a > 0$

dent, zero mean Gaussian white noise processes with constant variances σ_u^2 and σ_v^2, respectively; and $W(t) = e^{-at}$ is the system impulse response. The system equations are

$$\dot{x} = -a x + u(t)$$

$$z(t) = x(t) + v(t)$$

With $t_o = -\infty$, the optimal filtered estimate of x can be shown(1) to have a variance of

$$\bar{P} = \sigma_v^2 \left[\sqrt{a^2 + \frac{\sigma_u^2}{\sigma_v^2}} - a \right]$$

in the "steady-state".

For simplicity, it is assumed that $a = 1$, $\sigma_v^2 = 1$, and $\sigma_u^2 = 8$ from which it follows that $\bar{P} = 2$.

Now assume that it is desired to initiate smoothing at $t = 0$ to determine $\hat{x}(0/t)$, $t \geqslant 0$. For this example, Eq. (31) becomes

$$\dot{K}(t, 0) = - \left[-a + \frac{\sigma_u^2}{\bar{P}} \right] K(t, 0)$$

$$= -3 K(t, 0)$$

from which

$$K(t, 0) = e^{-3t} \qquad t \geqslant 0$$

recalling that $K(0,0) = 1$.

The smoothing error variance equation, Eq. (37), is then

$$\dot{P}(0/t) = -\frac{1}{\sigma_v^2} K^2(t,0)\, \bar{P}^2$$

$$= -4\, e^{-6t}$$

subject to the initial condition $P(0/0) = \bar{P} = 2$. By direct integration,

$$P(0/t) = \frac{4}{3}\left(1 + \frac{1}{2} e^{-6t}\right)$$

for $t \geqslant 0$. From this expression, it is seen that for one time unit of smoothing, the variance in the estimate of $x(0)$ is reduced from 2 to approximately 4/3.

Example 3. A simplified error model for the drift error in a single-degree-of-freedom gyroscope operating in a "zero-g" environment is given by the equations

$$\dot{\phi} = \epsilon + u(t)$$

$$\dot{\epsilon} = 0$$

where ϕ is the angular drift error in the pitch plane, ϵ is the "steady-drift" error rate, and $u(t)$ is the "random-drift" error rate. It is assumed that $u(t)$ is a zero mean, Gaussian white noise process with constant variance σ_u^2; and that $\epsilon(t_o)$ and $\phi(t_o)$ are independent, zero mean, Gaussian random variables with known variances. It is further assumed that the angular drift can be measured subject to Gaussian random errors. The measurement model is then

$$z(t) = \phi(t) + v(t)$$

where $v(t)$ is a zero mean, Gaussian white noise process with variance σ_v^2. From physical considerations, it is reasonable to assume that $v(t)$ is independent of $\phi(t_o)$, $\epsilon(t_o)$, and $u(t)$ for all t.

Letting

$$x = \begin{bmatrix} \phi \\ \epsilon \end{bmatrix}$$

It then follows that

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad H = [1 \quad 0]$$

$$Q = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \sigma_v^2$$

Now let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{12} \end{bmatrix}$$

be the covariance matrix of the <u>filtering</u> error where

$$p_{11}(t) = E\left\{[\phi(t) - \phi(t)]^2\right\}$$

$$p_{12}(t) = E\left\{[\phi(t) - \phi(t)]\ [\epsilon - \epsilon(t)]\right\}$$

and

$$p_{22}(t) = E\left\{[\epsilon - \epsilon(t)]^2\right\}$$

Substituting into Eq. (34), it is found that the <u>filtering</u> error variances (p_{11} and p_{22}) and covariance (p_{12}) are the solutions of

$$\dot{p}_{11} = 2\, p_{12} - \frac{1}{\sigma_v^2}\, p_{11}^2 + \sigma_u^2$$

$$\dot{p}_{12} = p_{22} - \frac{1}{\sigma_v^2}\, p_{11} p_{12}$$

$$\dot{p}_{22} = - \frac{1}{\sigma_v^2}\, p_{12}^2$$

For sufficiently long filtering time, the equilibrium solution is[1]

$$\bar{p}_{11} = \sigma_v \sigma_u \qquad \bar{p}_{12} = 0 \qquad \bar{p}_{22} = 0$$

Hence, by filtering alone, the steady drift error rate can be determined exactly, at least in theory.

Now suppose it is desired to initiate smoothing, after "steady-state" filtering has been attained, in an attempt to get an "improved" estimate of the drift angle ϕ. Since the steady drift error rate is known "exactly", its effect can be substracted out, and the system equations written

$$\dot{x} = u(t)$$

$$z(t) = x(t) + v(t)$$

where $x = \phi, t \geqslant T >> t_o$, and $P(T) = \bar{p}_{11} = \sigma_v \sigma_u$. Then, $F = 0$, $H = 1$, $Q = \sigma_u^2$, $P^{-1}(t) = 1/\bar{p}_{11} = 1/\sigma_v \sigma_u$, from which Eq. (31) becomes

$$\dot{K}(t, T) = -\frac{\sigma_u}{\sigma_v} K(t, T)$$

where $K(T, T) = 1$. Hence,

$$K(t, T) = e^{-\beta(t-T)} \qquad t \geqslant T$$

where $\beta = \sigma_u/\sigma_v$.

Equation (37) is

$$\dot{P}(T/t) = -\frac{1}{\sigma_v^2} \bar{p}_{11}^2 \quad K^2(t, T)$$

$$= -\sigma_u^2 \quad e^{-2\beta(t-T)}$$

and with $P(T/T) = P(T) = \sigma_v \sigma_u$, its solution is obviously

$$P(T/t) = \frac{\sigma_v \sigma_u}{2} \quad \left[1 + e^{-2\beta(t-T)}\right]$$

where $t \geqslant T$. For long smoothing times $t >> T$, it is seen that $P(T/t) \rightarrow \sigma_v \sigma_u/2$ which establishes the performance limit on the accuracy to within which the drift

angle ϕ can be determined.

It is remarked that these results are also valid if the gyroscope is torqued during smoothing, assuming that the torquing signal is known accurately. In this case, the torquing signal is input to the filter as a known forcing function.

Example 4. Consider the case where the state vector to be estimated is a set of n constants. Then, $\dot{x} = 0$ where it is assumed that $x(t_0)$ is a zero mean, Gaussian random n-vector having covariance matrix $P(t_o)$. Assume that the measurement model is

$$z(t) = H(t)\, x(t) + v(t)$$

where $H(t)$ is a real continous $m \times n$ matrix, and $v(t)$ is a zero mean, Gaussian white noise process with covariance matrix $R(t)$, and $E[x(t_o)\, v'(t)] = 0$ for all t.

Since $F(t) = Q(t) = 0$ for all t, the covariance equation for the filtering error, Eq. (34), is

$$\dot{P} = - P\, H'(t)\, R^{-1}(t)\, H(t)\, P$$

for $t \geq t_o$ with $P(t_o)$ given.

Now since x = constant, let $T = t_o$ for purposes of smoothing. Again, since $F(t) = Q(t) = 0$ for all t, Eq. (31) becomes

$$\dot{K}(t, t_o) = 0$$

from which it follows that $K(t, t_o) = I$ for all $t \geq t_o$ since $K(t_o, t_o) = I$. As a result, Eq. (37) assumes the form

$$\dot{P}(t_o/t) = - P(t)\, H'(t)\, R^{-1}(t)\, H(t)\, P(t)$$

for $t \geq t_o$ where $P(t_o/t_o) = P(t_o)$.

Obviously, the two covariance equations for filtering and smoothing are identical. It is trivial to show that the two filters are also identical in this case.

6.0 Conclusion

The filter and error covariance equations for optimal fixed-point continous linear smoothing have been developed by considering the limiting case of the results for the discrete smoothing problem. The procedure is of use in data filtering

problems where a smoothed estimate of the state of a continous linear system is desired at some specific time during the system's operation, and can be used "on-line" for that purpose.

The four simple examples were presented to indicate the nature of the results to be expected in applications. It was shown in these examples that there exist cases where little or no improvement can be obtained by fixed-point smoothing which contradicts the intuitive notion that "improved estimates can always be obtained by smoothing." In other cases, it was shown that significant improvement can be obtained.

The stability of the smoothing filter, Eq (27), and of the gain, Eq. (31), has not been investigated and is left as a problem for future study.

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13. ABSTRACT

The filter and error covariance equations for optimal fixed-point smoothing for continuous linear systems are developed. The development is carried out by considering the limiting case of the results for the same problem for discrete linear systems. The procedure is of use in estimation problems where a smoothed estimate of a continuous linear system's state is desired at some specified critical time during the system's operation. Four examples are presented to illustrate the results.

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